Maximal Tori in Compact Semisimple Lie Groups

Venkateshan K

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1 Introduction

In the theory of Lie Groups, a special part is played by the compact semisimple Lie Group. Many of the important properties of these groups arise because of the existence of maximal connected-abelian groups also called maximal torus. A very important theorem which is the main goal of this note states that every element of the Group sits in some maximal torus and that all maximal tori are conjugate to each other. We will give a full proof of this here including all the non-trivial results that are used.

2 Basic Definitions

Semisimple Lie Group: A Lie group G is semi-simple if its Lie Algebra LH has no abelian ideals, i.e there is no abelian subspace LH such that if X $\in LH$ then $[X, Y] \in LH\ \forall Y \in LG$.
Torus:A torus of a group G is a subgroup J that it is isomorphic to $\mathbb{T}^k$ where $\mathbb{T}$ is the factor group $\mathbb{R}/\mathbb{Z} = \{e^{2\pi i\theta} \mid 0 \leq \theta < 1\}$ and $k \in \mathbb{Z}^+$.
Maximal Torus: A torus T is said to be maximal if, given another torus T' with T $\subset$ T' then T=T'.
Center: The center Z(G) of a Group G is $\{x \in G \mid xy = yx\ \forall y \in G\}$
Generator : An element x $\in T$ is called a generator of the torus T if the set $\{x^n \mid n = 0\,, 1\,, \ldots\}$ is dense in group manifold of T.

3 Assumed Results

3.1

Image of a compact set under a continuous map is compact.

3.2

Given an infinite sequence $\{x_n\} \subset A$ and A compact there is a always convergent subsequence $x_{n_k} \to x$.

3.3

If a set $A \subset Y$ and Y is closed then $Closure(A) = \bar{A} \subset Y$

3.4

If the matrix of the adjoint representation Ad:$G \to$ Aut(LG) given by Ad$(x)Y = xYx^{-1}$is unitary then the matrix representation of ad:$LG \to$ LG defined by $\text{ad}_X(Y) = [X, Y]$ is anti-hermitian.

3.5

If α is irrational then the set $\{e^{i2\pi n\alpha} | n \in \mathbb{N}\}$ is dense in the unite circle $|z| = 1$.

3.6

If G is a subgroup of $(\mathbb{R}^n, +)$ and there is an open set F around the origin such that $F \cap G = 0$ then G is a lattice subgroup of $\mathbb{R}^n$ given by

$$G = \{\sum_{i=1}^{n} a_i \mathbf{e}_i | a_i \in \mathbb{Z}\}$$

for some basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ in $\mathbb{R}^n$

4 Required Theorems

4.1

The center of a compact semi-simple Lie Group is finite.

First we show that Z(G) is a closed set. Consider the map $\psi : G \to Aut(G)$ given by
$\psi(g)(h) = ghg^{-1}$.
Clearly Ker(ψ)=$\{g \in G \,| ghg^{-1} = h \,, \forall h \in G\}$= Z(G). Note that ψ is a continuous map and Ker(ψ) $= \psi^{-1}(e)$. Since inverse of a closed set is closed, Z(G) is closed. If Z(G) was a continuous group then its Lie Algebra $LZ(G)$ would be a subalgebra of LG. Also since Z(G) is an invariant abelian invariant subgroup, $LZ(G)$ would be an abelian ideal and thus G would not be semisimple. A contradiction [1]

[1] I still haven't ruled out the possibility of Z(G) being countably infinite. I dont have proof of this as of now but hopefully would be equipped with one during the presentation

4.2

Let G be a Lie Group and LG be its Lie Algebra. Then the set $W = \{e^{X_1}e^{X_2}\dots e^{X_k} | X_i \in LG;\ k = 1\dots\}$ *is a subgroup* G_0 *of G and that is precisely the identity-connected component of G.*

Proof: That G_0 is subgroup is obvious. To show that any identity-connected component can be written in that form, let us consider such an element $x \in G$ and let $\gamma : [0,1] \to G$ be a curve with $\gamma(0) = e$ and $\gamma(1) = x$. Then we can find $0 < t_1 < t_2 < \dots < t_n < 1$ such that $(\gamma(t_i))^{-1}\gamma(t_{i+1})$ in the neighborhood of identity and hence can be written as e^{X_i} for some $X_i \in LG$. So $x = e^{X_1}e^{X_2}\dots e^{X_n}$.

4.3

Let G be a connected Abelian Lie Group. Then G isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$

Proof Consider the map $\pi : LG \to G$ defined by $X \to e^X$. We first note that this map is onto. This follows from Theorem 1.1 by which we know that any element x in G is of the form $e^{X_1}e^{X_2}\dots e^{X_k}$ but since G is abelian all X_i's commute and $x = e^{X_1+X_2+\dots X_k} = e^X$ for some $X \in LG$. It can easily be seen that this map defines a homomorphism from (LG,+) to G. So, by first isomorphism theorem, $G \cong LG/Ker(\pi)$. From the fact that there is neighborhood N_g around $0 \in LG$ such that the map $X \to e^X$ is an homeomorphism (consequently one-one) $Ker(\pi) \cap N_g = \{0\}$. This implies(see 3.6) that the $Ker(\pi)$ is a discrete subgroup of LG . Since $LG \cong \mathbb{R}^n$, $Ker(\pi)$ is a lattice and hence can be written in the form $\{a_1X_1 + a_2X_2 + \dots a_kX_k \mid a_i \in \mathbb{Z}; k < n\}$ for some linearly independent $X_i's$ in LG. So Ker(π) $\cong \mathbb{Z}^k$ Now we can extend the set of $X_i's$ to form a basis in $\mathbb{R}^n$. So $LG/Ker(\pi) \cong \mathbb{T}^k \times \mathbb{R}^{n-k}$

4.4

Every torus T has a generator.

Proof : By Theorem 4.2 we can always find a basis $\{X_i \mid i = 1, 2\dots n\}$in the Lie Algebra LT of the Torus $\mathbb{T}^n$ such that $e^{a_1X_1+a_2X_2+\dots a_nX_n} = e$ iff $a_i's \in \mathbb{Z}$ Now if we choose $\alpha_1, \alpha_2, \dots \alpha_n$ such that they are linearly independent over rationals $\mathbb{Q}$, then $\{e^{n\sum_{i=1}^n \alpha_i X_i} \mid n = 1, 2\dots\}$ is dense in T (from 3.5) . So $y = e^{\sum_{i=1}^n \alpha_i X_i}$ is the generator of T.

4.5

5 Main Theorem

5.1

Given a compact semisimple Lie group G, there is exist maximal tori such that
(i) every element $x \in G$ is contained in some maximal torus.
(ii) Any two maximal tori are conjugate to each other.

5.2 Principal Lemma

Let G be a compact connected group and H a maximal torus in G. Define $p : G \times H \to G$ by $p(x, y) = xyx^{-1}$. Then

$$p(G \times H) = Im(p) = G \iff \bigcup_{x \in G} xHx^{-1} = G \tag{1}$$

We'll show that 5.1 follows from (1). For any $x \in G$ it follows from (1) that $x \in yHy^{-1}$ for some $y \in G$. If $yHy^{-1} \subset T$ for some torus T then $H \subset y^{-1}Ty$. It can easily seen that $y^{-1}Ty$ is a torus, but H being maximal

$$H = y^{-1}Ty \tag{2}$$

$$T = yHy^{-1} \tag{3}$$

implies yHy^{-1} maximal . So we have (i) of 5.1.
Consider a maximal torus T in G. Let the generator be y. Then from (1) $y \in xHx^{-1}$ for some $x \in G$. So

$$\{y^n\} \in xHx^{-1} \tag{4}$$

Since xHx^{-1} is closed, and $\{y^n\}$ is dense in T, by (3.3)

$$T \subset xHx^{-1} \tag{5}$$

But since T is maximal $T = xHx^{-1}$ which establishes (ii) of 5.1

Now, we need to prove the Lemma 1. We do this by induction on the dimension of the group. For n=1, H=G and Im(p)=G. Assume it is true for dim(G)<n. We will first prove the inductive step for semisimple Lie groups and then extend it to all compact connected groups.
Since G is semisimple the center $Z(G)$ is discrete. If we can show that

$$p[G \times (H \backslash Z(G))] = G \backslash Z(G) \tag{6}$$

then

$$G \backslash Z(G) \subset \text{Im}(p) \tag{7}$$

Clearly

$$\text{Closure of } G\backslash Z(G) = G \tag{8}$$

This implies (from (3.3))

$$G \subset \text{Im}(p) \subset G \tag{9}$$

(9) is possible only if G =Im(p) and we are through.
Now in order to establish (6) it is sufficient to show that $p[G \times H\backslash Z(G)]$ is both closed and open in $G\backslash Z(G)$.
For closure consider a converging sequence in p$[G \times (H\backslash Z(G)])$

$$g_n \rightarrow g \tag{10}$$

such that $g \in G\backslash Z(G)$. We need to show that $g \in p[G \times (H\backslash Z(G))]$.
Let $\{x_n\} \in G$ and $\{h_n\} \in H$ be such that $p(x_n, h_n) = g_n$ Since G is compact we have (from 3.2) a subsequence$\{x_{n_k}\}$ such that

$$x_{n_k} \rightarrow x \tag{11}$$

Again by compactness there exists a subsequence of $\{h_{n_k}\}$ such that

$$h_{n_{k_l}} \rightarrow h \tag{12}$$

Relabelling these sequences as x'_nandh'_n, we have from the continuity of p,

$$\text{p}(x'_n, h'_n) \rightarrow \text{p}(x, h) \tag{13}$$

$$\Rightarrow g'_n \rightarrow p(x, h) \tag{14}$$

By uniqueness of limits, (10) and (14)

$$g = p(x, h) \tag{15}$$

This proves that $p[G \times H\backslash Z(G)]$ is closed in $G\backslash Z(G)$. To prove that it is also open consider an element

$$g = p(x, y) = xyx^{-1} \tag{16}$$

Let us consider a small open neighborhood U of g not containing Z(G). We can always do this because Z(G) is discrete. Take any element $g' \in U$. Then

$$g' = gt = xyx^{-1}t \tag{17}$$

for some t in the neighborhood of identity. If $r = x^{-1}tx$ then

$$g = x(yr)x^{-1} \tag{18}$$

Now if there is an open neighborhood V in $G\backslash Z(G)$ around y it is clear we can choose U sufficiently small that $yr \in V$. In other words, in order to prove that for

every $g \in p[G \times (H \backslash Z(G))]$ there is an open neighborhood $U \subset p[G \times (H \backslash Z(G))]$ we need to just show that there exists an open neighborhood $V \subset p[G \times (H \backslash Z(G))]$ around every $y \in H \backslash Z(G)$.
Fix some $y_o \in H \backslash Z(G)$. Consider the normalizer

$$N(y_0) = \{x \in G | x y_0 x^{-1} = y_0\} \tag{19}$$

Let $N_1(y_0)$ be the identity-connected component of $N(y_0)$.
$N(y_0) \neq G$ because otherwise $y_0 \in Z(G)$.
Since G is connected

$$\dim(N(y_0)) < \dim(G) \tag{20}$$

For any $c \in H$, $c \in N(y_0)$ and in fact, since H is connected $c \in N_1(y_0)$. This implies

$$H \subset N_1(y_0) \tag{21}$$

Also H is the maximal torus in $N_1(y_0)$ From induction hypothesis (1), and (20)

$$p[N_1(y_0) \times H] = N_1(y_0) \tag{22}$$

Note that we are making use of the induction result for general compact connected groups . We now claim

$$p[G \times H] = p[G \times N_1(y_0)] \tag{23}$$

From (21) we know that $p[G \times H] \subset p[G \times N_1(y_0)]$. Consider an aritrary point $xnx^{-1} \in p[G \times N_1(y_0)]$. From (22) we know

$$n = n_1 l n_!^{-1} \tag{24}$$

for some $n_1 \in N_1(y_0)$ and $l \in H$.So,

$$xnx^{-1} = (xn_1) l (xn_1)^{-1} \in p[G \times H] \tag{25}$$

proving (23). So we have to show $p[G \times N_1(y_0)]$ contains a neighborhood of y_0.
Let us consider the subspace of the Lie Algebra generated by the map $p(e, y) = y$ in the neighborhood of $y = y_0$. Since it is an identity map the Lie algebra here is just the Lie Algebra

$$LN_1(y_0) \tag{26}$$

of the Lie-Subgroup $N_1(y_0)$.
Now look at the map $p(x, y_0) = x y_0 x^{-1}$. For any $X \in LG$ consider

$$f_X(t) = e^{tX} y_0 e^{-tX} \tag{27}$$

$$\frac{d}{dt} f_X^{-1}(0) f_X(t)|_{t=0} = y_0^{-1} X y_0 - X = \mathrm{Ad}(y_0)(X) - X \tag{28}$$

where Ad is the adjoint representation of the group. We choose the inner product on LG such a way that Adjoint representation is unitary. So the tangent subspace obtained here is

$$\bigcup_{X\in LG} \frac{d}{dt} f_X^{-1}(0) f_X(t)|_{t=0} = (\mathrm{Ad}(y_0) - 1)[LG] \tag{29}$$

Now for any $X \in LN_1(y_0)$, $e^{tX} y_0 e^{-tX} = y_0$

$$\Rightarrow y_0^{-1} X y_0 = X \tag{30}$$

$$\Rightarrow \mathrm{Ad}(y_0)(X) = X \tag{31}$$

If $\mathrm{Ad}(y_0)(X) = X$ then $e^{tX} y_0 e^{-tX} = y_0 \;\Rightarrow$

$$X \in LN_1(y_0) \tag{32}$$

From (31) and(32) we get

$$LN_1(y_0) = \{X \in LG | \mathrm{Ad}(y_0)(X) = X\} \tag{33}$$

$\Rightarrow$ $\mathrm{Ad}(y_0)$ leaves $LN_1(y_0)$ invariant. By unitarity it also leaves the orthogonal subspace $LN_1(y_0)^{\perp}$ invariant. From (33) we know

$$(\mathrm{Ad}(y_0) - I)[LG] = (\mathrm{Ad}(y_0) - I)[LN_1(y_0) \oplus LN_1(y_0)^{\perp}] = (\mathrm{Ad}(y_0) - I)[LN_1(y_0)^{\perp}] \tag{34}$$

Let C = $(\mathrm{Ad}(y_0) - I)_{LN_1(y_0)^{\perp}}$, i.e that block of the matrix which acts on $LN_1(y_0)^{\perp}$. We claim that

$$\det(\mathrm{C}) \neq 0 \tag{35}$$

Assume otherwise. Then $\mathrm{Ad}(y_0)_{LN_1(y_0)^{\perp}}$ has an eigenvector $W \in LN_1(y_0)^{\perp}$ with eigenvalue 1. But this would imply that $W \in LN_1(y_0)$ by (33). A contradiction . From (34) and (35) we infer that

$$(\mathrm{Ad}(y_0) - I)[LG] = [LN_1(y_0)^{\perp}] \tag{36}$$

Combining (26) and (36) we get that the Lie Algebra generated by the map p is all of LG. This implies for we can always find a neighborhood $S \subset G \times (H \backslash Z(G))$ whose image is a neighborhood of y_0 as was required after (18). This completes the proof for connected semisimple compact Lie Groups.

To extend this result to a general compact connected Lie Group let D be such a group and LD be its Lie Algebra. Define the inner product on LD in such a way that the Adjoint representation of D is diagonal.Define a subspace L_A by

$$L_A = \{X \in LD | [X, Y] = 0\,,\ \forall Y \in LD\} \tag{37}$$

Consider the orthogonal complement $L_A^\perp$ of LA in the chosen inner product. We claim that $L_A^\perp$ is a Invariant SubAlgebra (Ideal) of LD. Take any $X \in LA^\perp$ and $Y \in LD$. For all $Z \in LA$ we have

$$\langle Z|[X,Y]\rangle = \langle Z|\mathrm{ad}_X(Y)\rangle = -\langle \mathrm{ad}_X(Z)|Y\rangle = -\langle [X,Z]|Y\rangle = 0 \tag{38}$$

where we have used (3.4) in the penultimate step and (37) in the last step. This implies $[X,Y] \in LA^\perp$, the required condition for it to be an ideal. Let M and N be the two subgroups of D generated by LA and $LA^\perp$. By definition N is semisimple. Define a map By definition N is semisimple.

$$f : M \times N \to D \tag{39}$$

given by f(x,y)=xy. Clearly f is surjective. Let H be a maximal torus in D. Since $M \subset Z(D)$ MH is also a torus. But since H is maximal MH=H $\Rightarrow M \subset H$. Let $f^{-1}(H)$ be $M \times T$. It is easy to see that T is abelian. Since H is connected T must also be connected and hence a torus. Moreover if $T \subsetneq T' \subset N$ where T' is a torus then $H' = f[M \times T'] \supsetneq H$ is also a torus. But since H is maximal, this is not possible $\Rightarrow$ T is maximal in N. Since by our proof for semisimple Groups $\bigcup_{x\in N} xTx^{-1} = N$ we have

$$\bigcup_{x\in N} xHx^{-1} = f(\bigcup_{x\in N} x(M\times T)x^{-1}) = f(M \times \bigcup_{x\in N} xTx^{-1}) = f(M\times N) = D \tag{40}$$

This proves (1) for a general compact connected Lie Group.

References

[1] *Barry Simon.* Representations Of Finite and Compact Groups