DIFFERENTIAL GEOMETRIC APPROACH TO HAMILTONIAN MECHANICS

Course Project: Classical Mechanics (PHY 401)
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1. Introduction

Hamiltonian mechanics is geometry in phase space. It deals with an even dimensional manifold, a ‘phase space’, a symplectic structure and a function which is referred to as the Hamiltonian. Using such an approach, a formulation of mechanics can be obtained which is invariant under group of symplectic diffeomorphisms. When formulated with differential geometric concepts, many developments in mechanics can be simplified and understood properly. Also many abstract ideas of geometry arose in the study of mechanics. E.g. the concept of cotangent bundles. The Hamiltonian point of view allows us to solve completely a series of mechanical problems which do not yield solutions by other techniques.

2. Mathematical Definitions

2.1 Topological Spaces, Charts and Manifolds

- A topological space denoted by S has the following properties:

It has a collection of subsets, denoted by T.
Both the null set and S are in T.
Union of finite number of sets in T, belongs to T.
Intersection of finite number of sets in T, belongs to T.

- Any charts defined on a topological space S consists of an open set U, together with one-one mapping $\varphi : U \to \varphi(U) \subset R^n$

- A real n dimensional manifold, M is defined as a space with finite or countable collection of charts such that every point belonging to this space is represented in at least one chart.

An N-dimensional manifold is a space which is like $\mathbf{R}^n$ locally.

Example: The 2-sphere S = $\{\mathbf{x} \in \mathbf{R}^3 \mid \|\mathbf{x}\|^2 = 1\}$ is $\mathbf{R}^2$ locally.

2.2 Tangent Spaces and tangent bundles

- A tangent vector X_P at a point P on a manifold M is an ordered pair (X, P) both of which are n-tuples. X may be regarded as an ordinary vector and P the position vector of the foot of X.
- The set of all possible tangent vectors at a point P on a manifold M is known as the tangent space at P. (denoted by T_PM)
- On the manifold M, the collection of all tangent vectors X_P from points P in M is referred to as the tangent bundle TM.

On similar lines, cotangent vectors/spaces may be defined as dual vectors/spaces of the above. A dual vector being a linear functional that maps every vector to a member of **R**.

The notation for cotangent to T_PM being $T_P{}^*M$.

2.4 Exterior Forms

- A 1-form is a linear function $\omega : R^n \to R$.

$$\omega(\lambda_1\xi_1 + \lambda_2\xi_2) = \lambda_1\omega(\xi_1) + \lambda_2\omega(\xi_2)$$
$$\lambda_1, \lambda_2 \in R$$
$$\xi_1, \xi_2 \in R^n$$

- An exterior k-form is a function of k vectors which is k-linear and antisymmetric.
$$\omega(\lambda_1\xi_1 + \lambda_2\xi_1^{'}, \xi_2, \ldots\ldots\xi_k) = \lambda_1\omega(\xi_1, \ldots\ldots\xi_k) + \lambda_2\omega(\xi_1^{'}, \ldots\ldots\xi_k)$$

- The exterior product $\omega_1 \wedge \omega_2$ on pair of vectors $\xi_1 \wedge \xi_2 \in R^n$ is the oriented area of the image of the parallelogram with the sides $\omega(\xi_1)$ and $\omega(\xi_2)$.
$$\omega_1 \wedge \omega_2\ (\xi_1, \xi_2) = \omega_1(\xi_1)\omega_2(\xi_2) - \omega_1(\xi_2)\omega_2(\xi_1)$$

2.5 Differential Forms

- A differential 1-form is a map from tangent bundle TM to **R**.
$$\omega : TM \to R$$
$$\omega(X_P) \in R$$

Example: Let (U,φ) be a local chart of M, φ = (x^1,…, x^n). Every differential 1-form ω on U can be written uniquely as $\omega = \sum_{i=1}^{n} a_i dx_i$

- A differential 2-form is a bilinear, antisymmetric map from TM×TM to **R.**
$$\omega(x, y) \in R$$

This can be expressed uniquely as $\omega = \sum_{i \neq j} a(x) dx_i \Lambda dx_j$

2.6 Symplectic Structures

- A symplectic structure on 2n dimensional manifold, M is a closed, nondegenerate differential 2-form on M such that
$$\mathrm{d}\,\omega^2 = 0$$
$$\forall \xi \neq 0 \exists \eta : \omega^2(\xi, \eta) \neq 0$$

Example: In R^2, $\omega^2 = \sum dp \wedge dq$.

2.7 Lie Groups and Algebra

A Lie group is a group G which is a differentiable manifold such that group operations and inverse operations are differentiable.

A Lie algebra is formed by considering infinitesimal tangent space at the identity of lie group G. It follows the Jacobi identity.

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

3. Lagrangian Mechanics

Consider a differentiable manifold M and its tangent bundle TM, Lagrangian is defined as a differentiable map $L:TM \to R$. A map $\gamma : R \to M$ is a motion in Lagrangian system and L is a Lagrangian function if

$$\Phi(\gamma) = \int L(\gamma)dt$$

The most generalized formulation of a mechanical system is through the principle of least action or Hamilton's principle. This postulates the involvement of a function L. The motion of the system is described by the particular γ for which

$$S = \int_{t_1}^{t_2} L(\gamma)dt$$

is minimum. The integral S is refereed to as the 'action'.

The evolution of the local coordinates $q = (q^1 \ldots q^n)$ of a point $\gamma(t)$ under motion of Lagrangian motion on manifold M satisfies the Lagrange-Euler Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

The Lagrangian for a system of particles with positions (r_1, r_2 r_3 ……..r_n) is given by

$$L = \sum \frac{1}{2} m_a v_a^2 - U(r_1, r_2 ......r_n)$$

Upon application of a Legendre transformation to the Lagrangian, the resultant so obtained is called the Hamiltonian.

4. Hamiltonian Equations

The Legendre Transform takes a function $L:TM \to R$ defined over tangent bundle to a function $H: T^*M \to R$ on a cotangent bundle. The Lagrangian can be represented in terms of its dependencies as $L(q_i, \dot{q}_i, t)$.When a Legendre transformation is used to change variables from the derivatives of co-ordinates $\dot{q}_i$ to the momenta p_i; the resulting function is called the Hamiltonian.

$$H(p_i, q_i, t) = \sum_i \dot{q}_i p_i - L(\dot{q}_i, q_i, t)$$

The system of Lagrange's equations is equivalent to 2n first order equations

$$p' = -\left(\frac{dH}{dq}\right), q' = \left(\frac{dH}{dq}\right)$$

On a symplectic manifold (M, ω^2), Hamiltonian function H is a differentiable map
$H : M \to R$

5. Mathematical Model for a Mechanical System

- A configuration space M = M^N is a differentiable manifold of dimension N. Physically N is the degrees of freedom of the system.
- A symplectic structure on the phase space TM, tangent bundle of M.
- A hamiltonian structure on the phase space T*M, cotangent bundle of M.
- The Kinetic energy is a differentiable function T on TM, such that

$$T: TM \to R,\ T(v) = ½\langle v, v\rangle,\ v \in TM$$

- A force field is given by a 1-form $\omega = \sum_{i=1}^{N} F_i dx^i$ on M

6. Hamiltonian Mechanics:

6.1 Hamiltonian Vector Fields:

To each vector ξ, tangent to a symplectic manifold (M^{2n}, ω^2) at a point **x**, we associate a 1-form ω^1_ξ on TM_x by

$$\omega^1_\xi(\eta) = \omega^2(\eta, \xi) \quad \forall \eta \in TM_X \quad (1)$$

Consider an isomorphism I: $T^*M_X \to TM_X$ such that I preserves symplectic structure.
If H is a function on a symplectic manifold M^{2n}, dH is a differential 1-form on M, and at every point there is a tangent vector to M associated with it, defined above(1). H is known as Hamiltonian function and I dH is known as a hamiltonian vector field.

6.2 Hamiltonian phase flows:

Consider H: $M^{2n} \to R$ a hamiltonian function on a symplectic manifold M^{2n}. Assuming that field I dH corresponding to H gives group of diffeomorphisms $g^t : M^{2n} \to M^{2n}$,then

$$\frac{d}{dx} g^t x = IdH \quad \text{At } t = 0$$

Then g^t is called the Hamiltonian phase flow.

6.3 Law of conservation of energy
The function H is a first integral of the hamiltonian phase flow with hamiltonian function H.

Consider the derivative of H in the direction of vector η is equal to the value of dH on η. But η = I dH from the definition.

then eq.(1) yields dH(η) = $\omega^2(\eta$, IdH) = $\omega^2(\eta,\eta) = 0$

This shows that H is a first integral of hamiltonian phase flow.

6.4 Poisson Brackets:

Consider a function f(p,q,t), then its total time derivative is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum \left(\frac{\partial f}{\partial q_k} q'_k + \frac{\partial f}{\partial p_k} p'_k\right)$$

Poisson bracket of the quantities H and f is given by

$$[H,f] = \sum_k \left(\frac{\partial H}{\partial p_k}\frac{\partial f}{\partial q_k} - \frac{\partial H}{\partial q_k}\frac{\partial f}{\partial p_k}\right)$$

The functions of the dynamical variables which remain constant during the motion of the system are known as **integrals of the motion**.

6.5 Definition of Poisson bracket from differential geometric point of view:
The poisson bracket of the functions F and H given on a symplectic manifold is the derivative of the function F in the direction of the phase flow with hamiltonian function H.

$$[H, F](\mathbf{x}) = \frac{d}{dt} F(g_H^t(x)) \quad \text{at } t = 0$$

The Jacobi Identity: The poisson bracket of three functions f, g and h satisfies the Jacobi identity:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

Example: Consider the angular momentum of a particle **M = r × p,**

$[M_X, M_Y] = - M_z$, $[M_Y, M_Z] = - M_X$,

$[M_Z, M_X] = - M_Y$

Consider **B** and **C** the hamiltonian fields with hamiltonian functions B and C. Then [**B, C**] vector field is hamiltonian and its Hamiltonian function is [B, C].

6.6 Lie Algebra of Hamiltonian fields:

The hamiltonian fields form subalgebra of the Lie algebra of all vector fields.

The first integrals of a hamiltonian phase flow form a subalgebra of the Lie algebra of all functions. The Lie algebra of hamiltonian functions can be mapped onto the Lie algebra of hamiltonian vector fields.

6.7 Canonical Transformation

A canonical transformation on a Hamiltonian system is a change of co-ordinates that while not necessarily preserving the original form of the Hamiltonian, maintains the relations of canonical co-ordinates:

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$$

(Hamilton's equations)

The conditions that on applying a general transformation of co-ordinates, $(q, p, t) \to (q', p', t)$, the Hamilton equations retain their form are derived from substituting the transformed co-ordinates and comparing with the original :

$$\left(\frac{\partial q'_m}{\partial p_n}\right) = -\left(\frac{\partial q_n}{\partial p'_m}\right)$$

$$\left(\frac{\partial q'_m}{\partial q_n}\right) = -\left(\frac{\partial p_n}{\partial p'_m}\right)$$

Time dependence of the co-ordinates can be incorporated into the Hamiltonian formalism by means of using time a canonical transformation such as:

$$q'(t) = q(t+\tau)$$
$$p'(t) = p(t+\tau)$$

The Hamiltonian acquires the form

$$H = \frac{1}{2}\sum_i p_i(t+\tau)q_i(t+\tau) + U(p_i(t+\tau), q_i(t+\tau), t)$$
$$= \frac{1}{2}\sum_i p'_i(t)q'_i(t) + U(p'_i(t), q'_i(t), t)$$

More general definition of canonical transformations:
Consider a differentiable mapping of the phase space R^{2n} (**p, q**) to R^{2n}.
The mapping g is said to be a canonical transformation, if it preserves 2-form

$$\omega^2 = \sum dp_i \Lambda dq_i$$

6.8 Geometric interpretation of the Hamilton-Jacobi equation

1) Conservative systems: Hamiltonian $H : T^*M \to R$ and the real valued "action" $S : M \to R$.

$$dS : M \to T^*M\ ,\ x \to (x, d_x S)$$

The composite of these maps gives Hamilton-Jacobi equation on M, $H \circ dS$ = E, above equation implies

$$H(q_1, ..., q_n, \frac{\partial S}{\partial q^1}, ..., \frac{\partial S}{\partial q^n}) = E$$

2) Non-conservative systems: If H is the hamiltonian of a dynamical system, then

$$\frac{\partial S}{\partial t} + H(q_1, ..., q_n, \frac{\partial S}{\partial q^1}, ..., \frac{\partial S}{\partial q^n}, t) = 0$$

7. References:

[1] V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, 1978

[2] Landau and Lifshitz, Mechanics, Vol.1 of theoretical course, Pergamon Press, 1960

[3] Von Westenholz, Differential Forms in Mathematical Physics, North-Holland Publishing Company, 1981